Experiments with the Linear Automata and Synthesis Test to Them

Dmitry Speranskiy

Abstract - **The overview of results in area of the experiments theory with linear automata is given. This theory is a fundamental base to devise methods of discrete systems technical diagnosis.**

Index Terms – **automata, discrete system, technical diagnosis.**

I. Introduction

The mathematical model of processes and devices, named the finite–state machines (automaton), is a very simple one, however, its model is very convenient and being widely used in informatics and engineering. The theory of the finite–state machines is a fundamental unit of the modern informatics, but the theory of experiments with automata has a direct connection to the reliability problem of discrete devices.

An automaton is considered as a system with unknown internal structure but we are able to observe "external" behavior of automaton (response of automaton on input sequence).

Some results of research in theory of experiments with finite-state machines up to 1960s were sum up by *A. Gill* in [1].

According to A. Gill, experiment is a process of applying input sequences to automaton, observation of resultant output sequences and conclusions, based on those observations.

One of the central questions in the theory of experiments is how to find an input sequence for experiment process. It is shown in [1] that for the most general model (Mealy automaton) the construction methods of above mentioned input sequences are very labor-consuming.

One of possible ways to reduce the complexity is the way of research of particular automaton class. In our article the results for linear automata (LA) are represented. The specific character of *LA* simplifies the method of experiment construction and significantly decreases the experiment length.

It is important to notice that *LA* is an adequate model of many processes and devices in real life, e.g., devices for encoding and decoding of information process, signature analysis process, multiplication and division of binary polynomials can be defined by *LA* models.

Author of this article was involved in research of experiment automata theory for a long time. Some of our results we shortly introduce here but others were published in [3].

II. Basic definitions

Let's begin with the description of LA model [2]. *LA* is a system with finite number *l* and *m* of input and output poles, respectively. Input signals apply to all inputs in discrete time moments simultaneously. It is assumed that input signals are values from the field $GF(p) = \{0,1,..., p-1\}$, where *p* is a prime number.

LA state is an ordered set of the element delay states, which are part of LA structure. Let the number of such delays is *n*. The number *n* is called *LA* dimension and state set of *LA* is designated as S_n.

Let's introduce the following notations:

$$\overline{u}(t) = [u_1(t),..., u_l(t)]', \; \overline{y}(t) = [y_1(t),..., y_m(t)]',$$

$$\overline{s}(t) = [s_1(t),..., s_n(t)]'.$$

Here $\overline{u}(t)$, $\overline{y}(t)$, $\overline{s}(t)$ are input, output and vector–state respectively and *t* is a discrete time moment.

The functioning of *LA* is given by a system of equations of state and output respectively:

$$\overline{s}(t+1) = A\overline{s}(t) + B\overline{u}(t), \qquad (1)$$

$$\overline{y}(t) = C\overline{s}(t) + D\overline{u}(t), \qquad (2)$$

where $A = [a_{i,j}]_{n\times n}$, $B = [b_{i,j}]_{n\times l}$, $C = [c_{i,j}]_{m\times n}$, $D = [d_{i,j}]_{m\times l}$ are called characteristic matrices. Every matrix consists of the elements of $GF(p)$.

Using the mathematical induction method, we can prove that the final state and output response on input sequence $\overline{u}(0), \overline{u}(1),..., \overline{u}(k)$, of the length *k*–1, can be calculated by the following formulas, where $\overline{s}(0)$ is an initial *LA* state:

$$\overline{s}(k+1) = A^{k+1}\overline{s}(0) + A^k B\overline{u}(0) + A^{k-1}B\overline{u}(1) + \\ + ... + AB\overline{u}(k-1) + B\overline{u}(k), \qquad (3)$$

$$\overline{y}(k) = CA^k\overline{s}(0) + CA^{k-1}B\overline{u}(0) + CA^{k-2}B\overline{u}(1) + . \\ + ... + CB\overline{u}(k-1) + D\overline{u}(k). \qquad (4)$$

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Dmitry Speranskiy is with Saratov State University, *E-mail*: SperanskiyDV@info.sgu.ru

Now we shall define various types of experiments which will be used in our research. To keep it more compact, we shall make it with regard to the general model of Mealy automaton.

Mealy automaton is a set of five objects

$$A = (S, X, Y, \delta, \lambda),$$

where *S, X, Y* are finite sets of the states, input and output alphabets respectively, but $\delta : S \times X \to S$ and $\lambda : S \times X \to Y$ are the maps. These maps are called transition and output functions. Let S is set $S = \{x_1, ..., x_n\}$ and X is set $X = \{x_1, ..., x_l\}$.

Definition 1. The input sequence $p = x_{i_1}, x_{i_2}, ..., x_{i_a}$ is called a synchronizing sequence (*SS*) if $\forall s_{j1}, s_{j2} \in S \quad \delta(s_{j1}, p) = \delta(s_{j2}, p)$.

Definition 2. The input sequence $p = x_{i_1}, x_{i_2}, ..., x_{i_a}$ is called a homing sequence (*HS*) if $\forall s_{j1}, s_{j2} \in S$ $\lambda(s_{j1}, p) = \lambda(s_{j2}, p) \to \delta(s_{j1}, p) = \delta(s_{j2}, p)$.

It is obvious that *SS* is a singular *HS*, as applied *SS* leads *LA* to the known final state, however there is no need to observe automaton response.

Definition 3. The input sequence $p = x_{i_1}, x_{i_2}, ..., x_{i_a}$ is called a diagnostic sequence (*DS*) if $\forall s_{j1}, s_{j2} \in S$ $\lambda(s_{j1}, p) = \lambda(s_{j2}, p) \to s_{j1} = s_{j2}$.

It is clear that every *DS* is *HS*, at the same time, but the contrary statement is false.

III. Conditions of *SS*, *HS* and *DS* existence for *LA*

Let's note that conditions of existence of SS, HS and DS were defined in [1] but it has been made in terms of a complex construction of a successor-tree and therefore conditions verification is a very labor-consuming process.

The results given below will show that these conditions for LA can be easily verified.

Note that all statements listed in the article are given without proof. These proofs are done in the articles [4-10] and monograph [3] mentioned in References.

Theorem 1. A necessary and sufficient condition that *LA A* has *SS* of length *k* is $A^k = [0]$.

Here [0] is a null-matrix.

Theorem 2. If there is a certain SS of length *t* for LA then every input sequence of the same length or more is also SS for this LA.

Theorem 3. A length of minimal *SS* for *LA* of dimension *n* is not more than *n*.

This theorem provides a simple rule for *SS* existence verification: we need to exponentiation the matrix *A* k-times (k=2,3,…) until $A^k = [0]$. If $k \le n$ then SS exits, otherwise the process is stopped, *SS* does not exist for this *LA*.

Theorem 4. HS of the length *k+1* for LA $\tilde{A}$ exists if and only if $(\forall \bar{s} \in S_n \quad \bigvee_{d=0}^{k} CA^d \bar{s} \ne [0]) \vee A^{k+1} = [0]$.

Here $\bigvee_{d=0}^{k}$ is a conjunction of $(k+1)$ expressions after this sign.

Theorem 5. If *LA* $\tilde{A}$ with a nonsingular characteristic matrix *C* has at least one *HS* of length $(k+1)$ then all sequences of the same length and more are *HS*s for this *LA*.

It means that *HS* construction problem comes to a problem how to find a natural number *k* that *HS* of the length *k* for given *LA* exists. Note that the same problem for Mealy automaton is not trivial and required labor-consuming solution methods.

Theorem 6. A length of minimal *HS* for *LA* of dimension *n* is not more than *n*.

Now we shall consider *DS*.

Theorem 7. *DS* of the length *t* for *LA* $\tilde{A}$ of the dimension *n* exists if and only if the rank of matrix

$$K_t = \begin{bmatrix} C \\ CA \\ CA^2 \\ ..... \\ CA^{t-1} \end{bmatrix}$$

is equal to *n*.

Theorem 8. If LA has at least one DS of length *k* then any input sequence of length *k* and more will be also DS for this LA.

It was proved in [1] that in the general case the minimality of Mealy automaton is a necessary but not sufficient condition of DH existence for these automata. The following statement is true for LA.

Theorem 9. If *LA* is minimal automata then *DS* for this *LA* exists.

Corollary. A length of minimal *DS* for *LA* of the dimension *n* is not more than *n*.

Finding *DH* for given *LA* is called a diagnostic problem. It is known [1] that the ability to solve such problem depends on set of admissible initial states and used methods. It was shown in [1] that the most powerful method to solve the diagnostic problem is multiple experiments. Simple unconditional experiments are less helpful in this situation. The following statement shows that above mention is not true for *LA*.

Theorem 10. The diagnostic problem is always solvable by a simple unconditional diagnostic experiment for any minimal *LA* and for any set of admissible initial states.

IV. Experiments with the non-stationary ***LA***

Now we shall consider so-called non-stationary *LA* (*NLA*) that is described by system of equations

$$\bar{s}(t+1) = A(t)\bar{s}(t) + B(t)\bar{u}(t),$$

$$\bar{y}(t) = C(t)\bar{s}(t) + D(t)\bar{u}(t).$$

The matrix dimensions in these equations are the same as in (1) and (2).

It is proved by analogy with *LA* that the final state and the output response of *NLA* after application of input sequence $\bar{u}(0), \bar{u}(1), ..., \bar{u}(t)$ can be calculated by the following formulas:

$$\bar{s}(t+1) = A(t)A(t-1)...A(0)\,\bar{s}(0) +$$

$$+\sum_{i=0}^{t-1} A(t)A(t-1)...A(i+1)(B(i)\bar{u}(i) + B(t)\bar{u}(t),$$

$$\bar{y}(t) = C(t)A(t-1)...A(0)\,\bar{s}(0) +$$

$$+\sum_{i=0}^{t-1} C(t)A(t-1)...A(i+1)B(i)\bar{u}(i) + D(t)\bar{u}(t).$$

Here $\bar{s}(0)$ is the initial state of *NLA*.

The validity of below listed statements is proved in [3].

Theorem 11. The input sequence $\bar{u}(0), \bar{u}(1), ..., \bar{u}(t)$ is *SS* for *NLA* $\widetilde{A}$ if and only if

$$\forall\ \bar{s}_1(0), \bar{s}_2(0) \in Init(\widetilde{A})$$

$$A(t)A(t-1)...A(0)[\bar{s}_1(0) - \bar{s}_2(0)] = [0].$$

Here $Init(\widetilde{A})$ is admissible initial state set of *NLA* $\widetilde{A}$.

Corollary 1. If $Init(\widetilde{A}) = S_n$ then a necessary and sufficient condition of existence of *SS* of length $(t+1)$ for *NLA* $\widetilde{A}$ is the following one:

$$A(t)A(t-1)...A(0) = [0].$$

Corollary 2. If $Init(\widetilde{A}) = S_n$ and there is a certain *SS* of length *t* for *NLA* then every input sequence of the length *t* or more is also a *SS* for this *NLA*.

It is easy to see that in the general case the length of *SS* has no upper bound.

Let's consider *NLA* with the following characteristic matrices:

$$A(i) = E \text{ for } i = 0,1,...,t-1, A(t) = 0,$$

where *E* is init matrix. It is clear from Theorem 11 that such *NLA* has *SS* of length (*t*+1) but every input sequence of the length less than (*t*+1) is not *SS*. As far as *t* is an arbitrary parameter therefore above mentioned statement is true.

Now we shall consider the special class of *NLA*, so-called periodical *NLA*. Every such automaton has periodical characteristic matrices. In other words, there is integer positive number λ that $A(t+\lambda) = A(t)$, $B(t+\lambda) = B(t)$, $C(t+\lambda) = C(t)$, $D(t+\lambda) = D(t)$.

Now we construct a stationary automata $\widetilde{A}_{st}$ (based on *NLA* $\widetilde{A}$) with the following transition function:

$$\bar{s}(t+1) = \hat{A}\cdot\bar{s}(t),$$

where $\hat{A} = A(\lambda-1)A(\lambda-2)\ldots A(0)$.

Theorem 12. There is *SS* for periodical *NLA* $\widetilde{A}$ if and only if there is *SS* for the stationary automata $\widetilde{A}_{st}$.

Theorem 13. If λ is a period of characteristic matrix $A(t)$ of *NLA* $\widetilde{A}$ then the length of minimal *SS* is not more than λn, where *n* is *NLA* dimension.

Theorem 14. The input sequence $\bar{u}(0), \bar{u}(1), ..., \bar{u}(t)$ is *HS* for periodical *NLA* $\widetilde{A}$ if and only if

$$\forall\ \bar{s}_1(0), \bar{s}_2(0) \in Init(\widetilde{A}) \quad \exists k \in [0,t]$$

$$(C(k)A(k-1)...A(0)[\bar{s}_1(0) - \bar{s}_2(0)] \neq [0]) \vee$$

$$\vee\ (A(t)A(t-1)...A(0)[\bar{s}_1(0) - \bar{s}_2(0)] = [0])$$

Corollary. If there is a certain *HS* of length *t* for periodical *NLA* then every input sequence of the same length or more is also *HS* for this *NLA*.

By analogy with the stationary *LA* we introduce so-called diagnostical matrix of *NLA:*

$$K_t = \begin{bmatrix} C \\ CA(0) \\ CA(1)A(0) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ CA(t-1)A(t-2)...A(0) \end{bmatrix}.$$

Theorem 15. The input sequence $\bar{u}(0), \bar{u}(1), ..., \bar{u}(t)$ is *DS* for periodical *NLA* if and only if

$$rank\ K_t = n,$$

where *n* is *NLA* dimension.

Corollary. If there is a certain *DS* of the length *t* for the periodical *NLA* then every input sequence of the same length or more is also *DS* for this *NLA*.

V. *LA* TESTING

Faults can occur during exploitation of digital devices (*DD*) therefore its checks should be carried out. One of ways for fault detection is applying a special input sequence (test) to device. A response of *DD* on test must differ subject to a technical state of *DD* (good or faulty). Consequently, the fault detection process is an experiment with *DD*.

It is clear that experimenter must know the initial state of this *DD* before application of *DD* test. The identification of initial state in some cases may be carried out by *SS*, *HS* and *DS* that was described in 3rd section. The test construction methods that were known earlier contained hard restrictions with regard to *DD* structure and information about the initial state of *DD*.

The methods of the test construction suggested below don't demand to comply with mentioned restrictions and less labor-consuming.

Now we shall define a problem which is investigated in this section.

LA and its fault admissible modification are given. It is required to construct the input sequence (test) which will detect above mentioned fault. In other words, the response of good and faulty *LA* on test must be different regardless of initial states.

Let's remind some definitions we'll use further.

We say that *LA* $\widetilde{A}$ has a finite memory of depth μ if for any time moment *t* takes place the following equality:

$$\bar{y}(t) = f(\bar{u}(t), \bar{u}(t-1), ..., \bar{u}(t-\mu), \bar{y}(t-1), ..., \bar{y}(t-\mu))$$

It means that we can predict *LA* reaction in any time moment *t* if we know an input sequence and response of *LA* in previous μ time moments.

It is known [2] from *LA* theory that every *LA* has a finite memory of depth μ where $\mu \le n$ (*n* is *LA* dimension).

We say that *LA* is μ-definite one if *LA* response in discrete moment *t* depends only on previous μ inputs:

$$\bar{y}(t) = f(\bar{u}(t), \bar{u}(t-1), ..., \bar{u}(t-\mu))$$

Now pass on to description of test construction method. Let A_1, B_1, C_1, D_1 be the characteristic matrices of faulty *LA* $\widetilde{A}_1$.

Let's consider a case when good *LA* and a faulty one are both μ-definite ones but values of parameters μ are different. Let $\mu = \max(\mu_1, \mu_2)$ where $\mu_1(\mu_2)$ is a depth of memory for *LA* $\widetilde{A}(\widetilde{A}_1)$. It is proved in [2] that a necessary and sufficient conditions for μ-definiteness of *LA* is the equality $CA^{\mu} = [0]$ This implies that $CA^{k} = [0]$ and $C_1 A_1^{\,k} = [0]$ for any $k \ge \mu$. Take into account these equalities and (4), then the reaction on input sequences $\bar{u}(0), \bar{u}(1), ..., \bar{u}(\mu)$ both *LA*s, regardless of their initial states, should be

$$\bar{y}(\mu) = CA^{\mu-1}B\bar{u}(0) + CA^{\mu-2}B\bar{u}(1) + ... + CB\bar{u}(\mu-1) + D\bar{u}(\mu),$$

$$\bar{y}_1(\mu) = C_1 A_1^{\mu-1} B_1 \bar{u}(0) + C_1 A_1^{\mu-2} B_1 \bar{u}(1) + ... + D_1 \bar{u}(\mu).$$

Subtracting one form another, we get

$$\bar{y}(\mu) - \bar{y}_1(\mu) = [CA^{\mu-1}B - C_1 A_1^{\mu-1} B_1]\,\bar{u}(0) + ... + [D - D_1]\;\bar{u}(\mu). \quad (5)$$

It is clear that the given fault is detected by input sequence $\bar{u}(0), \bar{u}(1), ..., \bar{u}(\mu)$ (test) if $\bar{y}(\mu) - \bar{y}_1(\mu) \ne [0]$.

The relation (5) we shall interpret as a system of linear algebraic equations (*SLAE*) of unknown variables

$$u = [u_1(0), ..., u_l(0), ..., u_1(\mu), ..., u_l(\mu)]. \quad (6)$$

Let *Q* is the matrix of system (5), then (5) may be written in the following form:

$$Qu = y, \quad (7)$$

where *y* is a *m*-dimensional nonzero vector.

Let *T* is the set of all tests for detection of given fault. In order to find the set *T* we need to vary the right side of (7) and find solutions of corresponding system.

It is clear that the number of different nonzero vectors *y* is $p^{(\mu+1)m}$. Even for small values μ and *m* this value is very big. Therefore we shall consider more effective method.

We shall consider a homogeneous system instead of (7)

$$Qu = [0]. \quad (8)$$

Let U_0 is a set of solutions for this system. If *U* is a set of all vectors of type (6) then obviously a set $U \setminus U_0$ is a set *T*, so we reduce finding solutions to one homogeneous system (8).

Now we pass on to test construction for lock-in *LA*. We say that *LA* is a lock-in one if there is an *SS* for this *LA*.

Theorem 16. Any lock-in *LA* is a $\mu-$definite *LA* at the same time.

It is clear that application of above presented test construction method for the lock-in *LA* can be based on this theorem.

Now we shall describe test construction method for arbitrary *LA* but not for lock-in ones only. This method is based on the fact that any *LA* has a finite memory.

Let good (faulty) *LA* has the depth memory $\mu_1(\mu_2)$ and $\mu = \max(\mu_1, \mu_2)$. It is known from [2] that the output functions *LA* $\widetilde{A}$ and faulty *LA* $\widetilde{A}_1$ always can be presented in the following form:

$$\bar{y}(t) = V_0\bar{u}(t) + V_1\bar{u}(t-1) + ... + V_\mu\bar{u}(t-\mu) + W_1\bar{y}(t-1) + ... + W_\mu\bar{y}(t-\mu),$$

$$\bar{y}_1(t) = V_0^1\bar{u}(t) + V_1^1\bar{u}(t-1) + ... + V_\mu^1\bar{u}(t-\mu) + W_1^1\bar{y}(t-1) + ... + W_\mu^1\bar{y}(t-\mu), \quad (9)$$

where $V_i(V_i^1)$, $W_i(W_i^1)$ are matrices of corresponding dimension.

It is clear that if $\bar{u}(t-\mu), \bar{u}(t-\mu-1), \bar{u}(t)$ is test of the minimal length then $\bar{y}(t-j) = \bar{y}_1(t-j)$ for $j = 1, ..., \mu$, but $\bar{y}(t) \ne \bar{y}_1(t)$.

Subtracting one form from another in (9), we get

$$\bar{y}(t) - \bar{y}_1(t) = [V_0 - V_0^1]\bar{u}(t) + ... + [V_\mu - V_\mu^1]\bar{u}(t-\mu) + [W_1 - W_1^1]\bar{y}(t-1) + ... + [W\mu - W_\mu^1]\bar{y}(t-\mu). \quad (10)$$

Equating (10) to some nonzero vector, we get *SLAE* of unknown variables, which are the coordinate of vector

$$u = [u_1(t-\mu), ..., u_l(t-\mu), ..., u_1(t), ..., u_l(t)]'.$$

Let *Q* is the matrix of obtained *SLAE* then this *SLAE* can be written as

$$Qu = y. \quad (11)$$

Note that the finding solutions of system (11) is pretty the same as for system (7).

Thus, the construction of all test sets for arbitrary *LA*s, both μ - definite and lock-in ones, can be reduced to a solution of one homogeneous system of equations.

VI. Conclusion

The above represented results show that the specific of *LA*s significantly simplify the construction of experiment theory for them. Thus, this specific makes it possible to decrease the upper bound of length for arbitrary types of experiments in comparison to corresponding assessment that is known for Mealy automaton. In addition, mentioned specific allows to reduce the problem of the experiment construction (in the general case this problem is very complex and labor-consuming) to more simple existence problem for such experiments. The last problem is solved by simple calculation of multiplying matrices, matrix exponentiation or matrix ranks. In other words, the conditions of existences of experiments can be easily verified.

It should be noted that after experiment's finished the identification of state can be carried out by solution of *SLAE*. There are well known and good developed mathematical methods for that.

Moreover, it is significant that above described methods provide test construction of length $\mu + 1$, where μ is the depth of *LA* memory. Since, as it's well known [2], $\mu \leq n$, where *n* is *LA* dimension, the methods shown in this article can provide very short test.

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